

Motivation	Existing work	Proposed method	Conclusions
<ul style="list-style-type: none"> <li>Typical ROM methods apply <b>spatial projection</b></li> <li>Reduces spatial complexity</li> <li>Does not reduce the temporal complexity: large number of time steps</li> <li>Error bounds grow exponentially in time</li> </ul>	<ul style="list-style-type: none"> <li>Explicit time integration with larger time steps in ROMs [Krysl et al., 2001]</li> <li>Limited realizable speedup; not applicable to stiff systems</li> <li>Space-time finite-element full-order models [Yano, 2014]</li> <li>Error bounds grow linearly in time</li> <li>Require a different full-order-model formulation; not always practical</li> </ul>	<ul style="list-style-type: none"> <li><b>Main idea:</b> space-time projection to fully discrete, implicit nonlinear ODE models</li> <li><b>Projection:</b> Least-squares Petrov-Galerkin to space-time discrete residual</li> <li><b>Hyper-reduction:</b> Sample elements of the space-time discrete residual</li> <li><b>Tensor decomposition:</b> multi-array data decomposition</li> </ul>	<ul style="list-style-type: none"> <li>Complexity is independent of both space and time</li> <li>Slower time growth in error bound</li> <li>Amenable to any time integrator</li> <li>Showed a great potential of tensor decomposition</li> <li>Offline cost is expensive</li> </ul>

## Full-order model

- System of nonlinear ordinary differential equations (ODEs)

$$\frac{d\mathbf{w}}{dt} = \mathbf{f}(\mathbf{w}, t; \boldsymbol{\mu}), \mathbf{w}(0; \boldsymbol{\mu}) = \mathbf{w}_0(\boldsymbol{\mu})$$

where  $\mathbf{w} : [0, T] \times \mathbb{R}^{N_p} \rightarrow \mathbb{R}^{N_s}$  is the state,  $\mathbf{w}_0 : \mathbb{R}^{N_p} \rightarrow \mathbb{R}^{N_s}$  the initial condition,  $\mathbf{f} : \mathbb{R}^{N_s} \times [0, T] \times \mathbb{R}^{N_p} \rightarrow \mathbb{R}^{N_s}$  is a (non-)linear velocity, and  $\boldsymbol{\mu} \in \mathbb{R}^{N_p}$  is a parameter vector.

- Discrete residual at time step  $n$  (linear multistep schemes):

$$\mathbf{r}^n(\mathbf{w}) := \alpha_0 \mathbf{w} - \Delta t \beta_0 \mathbf{f}(\mathbf{w}, t^n, \boldsymbol{\mu}) + \sum_{j=1}^k \alpha_j \mathbf{w}^{n-j} - \Delta t \sum_{j=1}^k \beta_j \mathbf{f}(\mathbf{w}^{n-j}, t^{n-j}, \boldsymbol{\mu})$$

where  $\Delta t$  is the time step, the coefficients  $\alpha_j$  and  $\beta_j$  define a specific linear multistep scheme.

## Spatiotemporal trial subspace

- Solution approximation:

$$\tilde{\mathbf{w}} \in \underbrace{\mathbf{w}_{ref} \otimes \mathbf{1}_{N_t} + \bigoplus_{i=1}^{n_s} \mathcal{S}_i \otimes \mathcal{T}_i}_{\text{space-time subspace}} \subseteq \mathbb{R}^{N_s} \otimes \mathbb{R}^{N_t}$$

$$\tilde{\mathbf{w}} = \mathbf{w}_{ref} \otimes \mathbf{1}_{N_t} + \sum_{i=1}^{n_s} \sum_{j=1}^{n_t^i} (\psi_j^i \otimes \phi_i) \hat{w}_{ij}$$

- # degrees of freedom:  $\dim(\bigoplus_{i=1}^{n_s} \mathcal{S}_i \otimes \mathcal{T}_i) = \sum_{i=1}^{n_s} n_t^i$

## Spatiotemporal LSPG projection

- Space-time discrete-residual minimization

$$\hat{\mathbf{w}}_{st} = \arg \min_{\hat{\mathbf{y}}} \left\| \bar{G} \bar{r} \left( \mathbf{w}_{ref} \otimes \mathbf{1}_{N_t} + \sum_{i=1}^{n_s} \sum_{j=1}^{n_t^i} (\psi_j^i \otimes \phi_i) \hat{y}_{ij} \right) \right\|_2^2$$

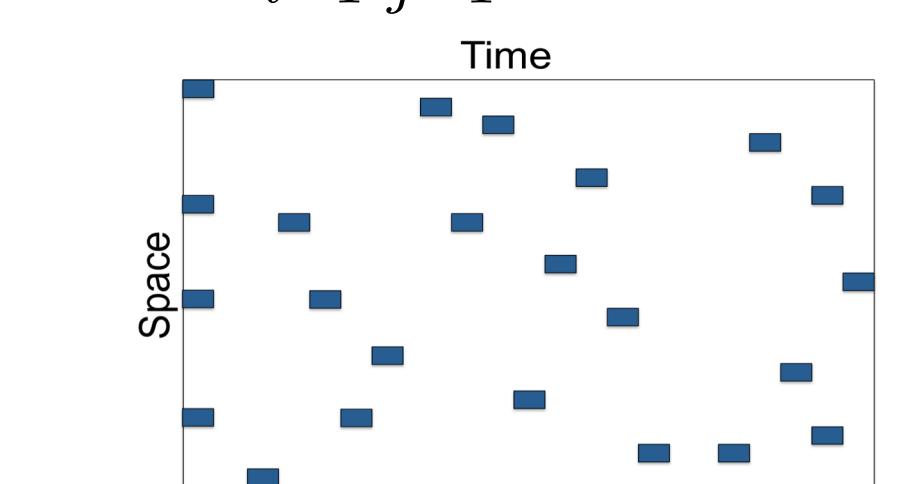
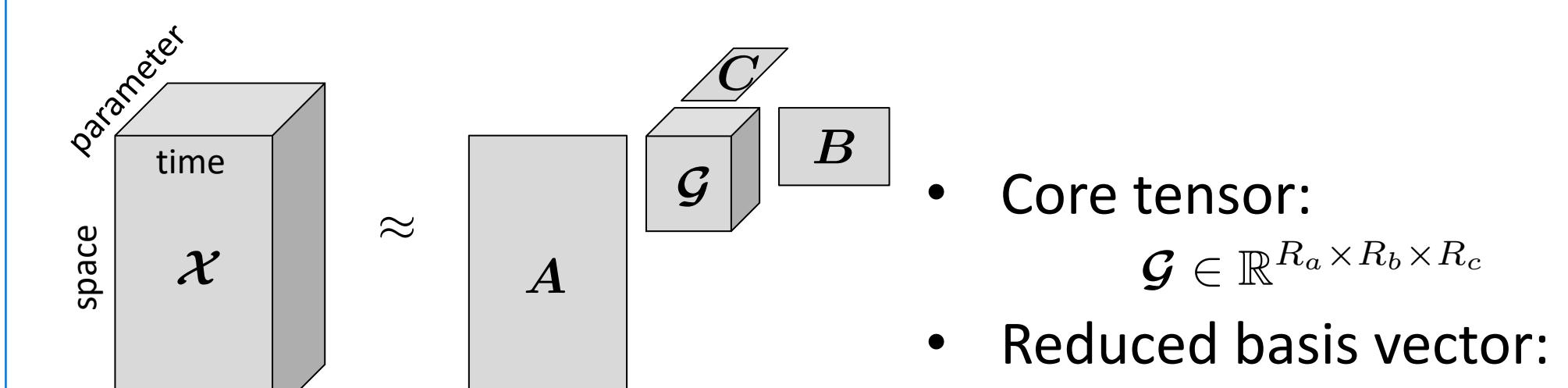


Figure 1. Spatiotemporal GNAT  $Z$

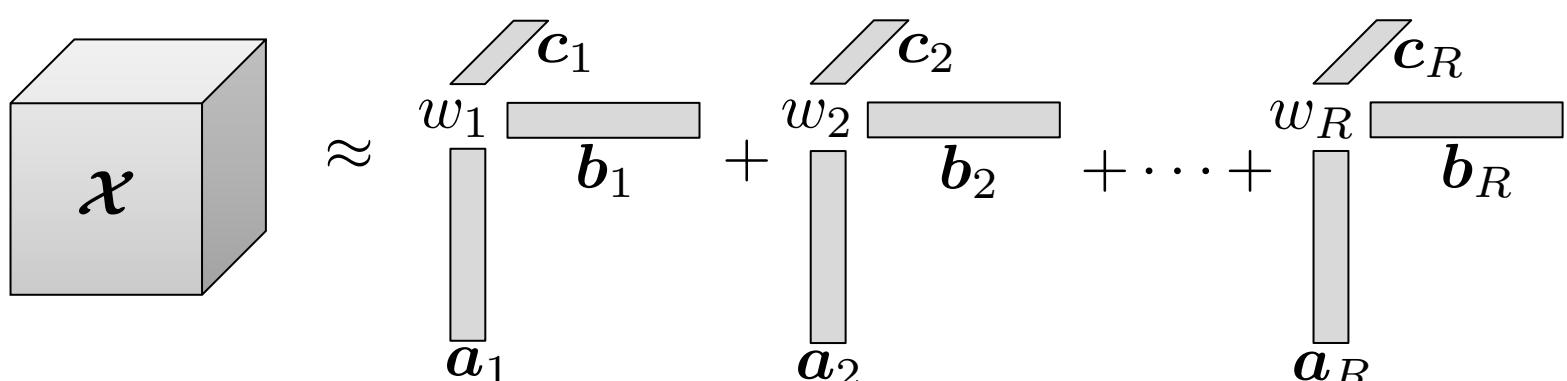
- ST-LSPG:  $\bar{G} = \bar{I}$
- ST-Collocation:  $\bar{G} = \bar{Z}$
- ST-GNAT:  $\bar{G} = (\bar{Z} \Phi_r)^\dagger \bar{Z}$

## Tucker decomposition



- Orthonormal factor matrices; Accuracy under control
- Guaranteed to be linearly independent
- Number of unknowns in ROM:  $R_a \times R_b \times R_c$
- It suffers from the curse of dimensionality

## Canonical Polyadic decomposition



- Reduced basis vector:  $\varphi_i = a_i \otimes b_i \otimes c_i$
- May result in linearly dependent basis
- Numerically unstable; hard to control accuracy
- Number of unknowns in ROM:  $R$
- Dose not suffer from the curse of dimensionality

## Spatiotemporal error bound

- Backward Euler time integration

$$\|\bar{w}_*^n - \bar{w}_{PG}^n\|_2 \leq \sqrt{N_t(1+\Lambda)} \min_{\bar{y} \in \mathcal{S}} \max_{k \in \mathbb{N}(N_t)} \|\bar{w}_*^k - \bar{y}^k\|_2$$

Λ grows polynomially in time with degree of 3/2

## References

- P. Krysl, S. Lall, and J.E. Marsden. Dimensional model reduction in non-linear finite elements dynamics of solids and structures. *Int. J. Numer. Math. Engng.*, 51(4):479-504, 2001.
- M. Yano. A space-time Petrov-Galerkin certified reduced basis method: application to the Boussinesq equations. *SIAM J. Sci. Comput.*, 36(1):A232-A266, 2013.
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## Proposed method

- Main idea:** space-time projection to fully discrete, implicit nonlinear ODE models
- Projection:** Least-squares Petrov-Galerkin to space-time discrete residual
- Hyper-reduction:** Sample elements of the space-time discrete residual
- Tensor decomposition:** multi-array data decomposition

## Numerical experiments

### 1D Burgers' equation

- Governing equations  $t \in [0, 0.5] \quad x \in [0, 1]$
- $\frac{\partial w(x, t)}{\partial t} + \frac{\partial f(w(x, t))}{\partial x} = 0.02 e^{u_{2x}}$
- $w(0, t) = \mu_1 \quad w(x, 0) = 1$
- Spatial discretization: a finite-volume method  $\Delta x = 10^{-2}$
- Time integrator: Backward-Euler method  $\Delta t = 2.5 \times 10^{-4}$
- Parameter:  $\boldsymbol{\mu} \in \mathcal{D} = [1.2, 1.5] \times [0.02, 0.025]$
- Training set:  $\mathcal{D}_{train} = \{1.2, 1.3, 1.4, 1.5\} \otimes \{0.02, 0.025\}$

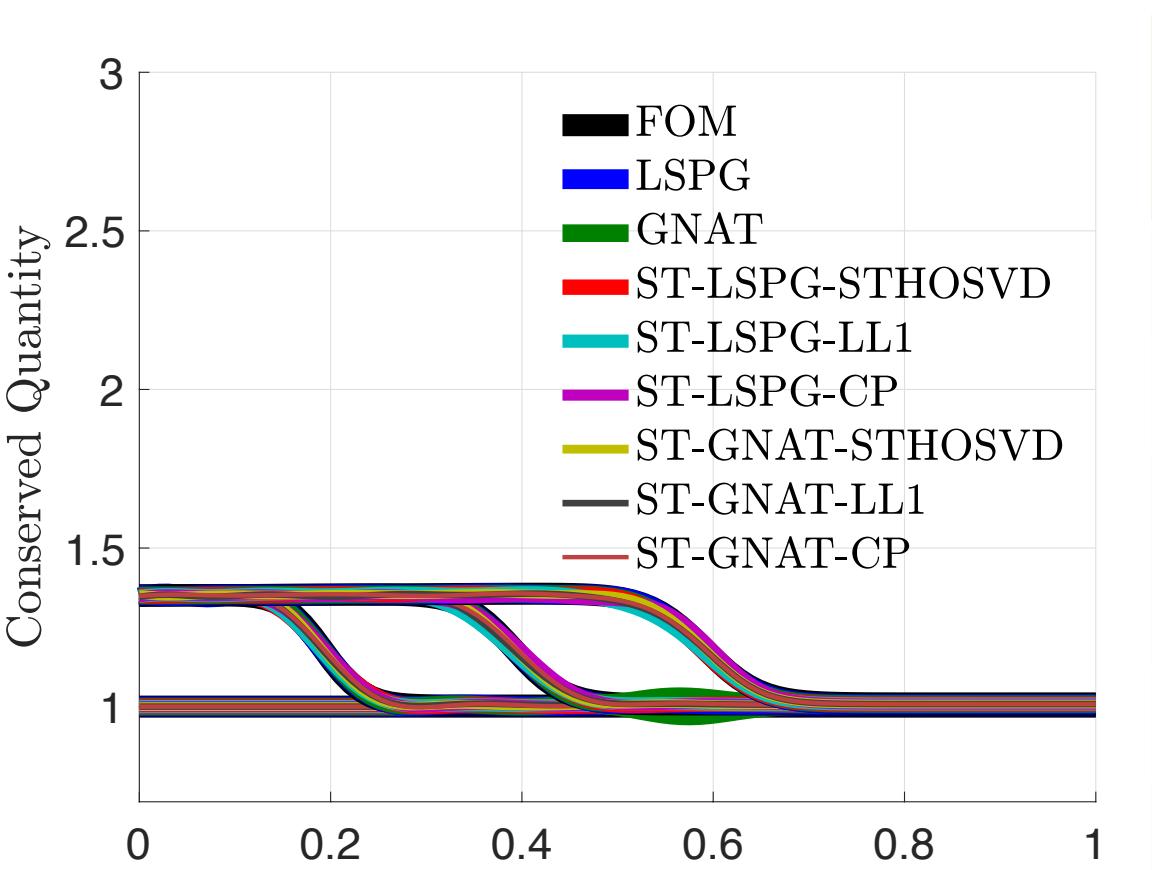


Figure 2.  $n_s = 15$ ,  $n_t = 20$ ,  $n_t^i = 2$   
 GNAT:  $n_z = n_r = 55$   
 GNAT-ST1:  $\bar{n}_s = 30$ ,  $\bar{n}_t = 120$ ,  $n_{r,s} = 100$ ,  $n_{r,t}^i = 3$   
 GNAT-ST2:  $\bar{n}_s = 30$ ,  $\bar{n}_t = 120$ ,  $n_{r,s} = 100$ ,  $n_{r,t}^i = 10$

Method	Rel. Error	Speedup
LSPG	0.27	0.82
GNAT	0.98	1.2
ST-LSPG-HOSVD	0.43	0.09
ST-LSPG-II1	0.51	0.67
ST-LSPG-CP	0.67	1.1
ST-GNAT-HOSVD	0.27	5.4
ST-GNAT-II1	0.40	15
ST-GNAT-CP	0.46	43

Table 1. Accuracy and speedup

### Boltzmann equation for the neutron flux

- Governing equations  $\frac{1}{\nu(E)} \frac{\partial \psi}{\partial t} + \Omega \cdot \nabla \psi + \sigma(r, E) \psi = \int_0^\infty \int_{4\pi} \sigma_s(r, E' \rightarrow E, \Omega' \cdot \Omega) \psi(r, E', \Omega', t) d\Omega' dE' + q(r, E, \Omega, t)$

- 17 energy group
- 16 angular directions
- 120 radial zones
- 40 time steps (0 ~ 1 μsec)
- Total degrees of freedom: 1,305,600

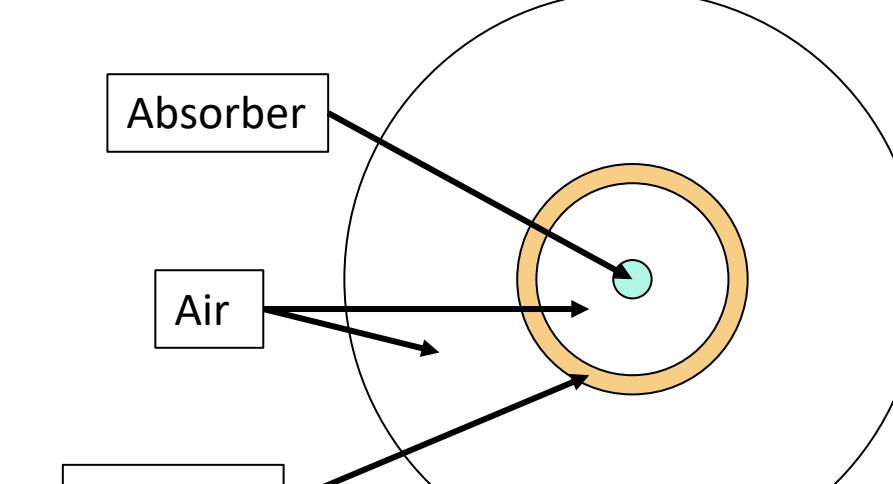


Figure 3. Geometry

Method	# DoFs	Rel. Error
LSPG	800	6.01e-6
ST-LSPG-II1	20	6.85e-7

Table 2. Number of DoFs and accuracy

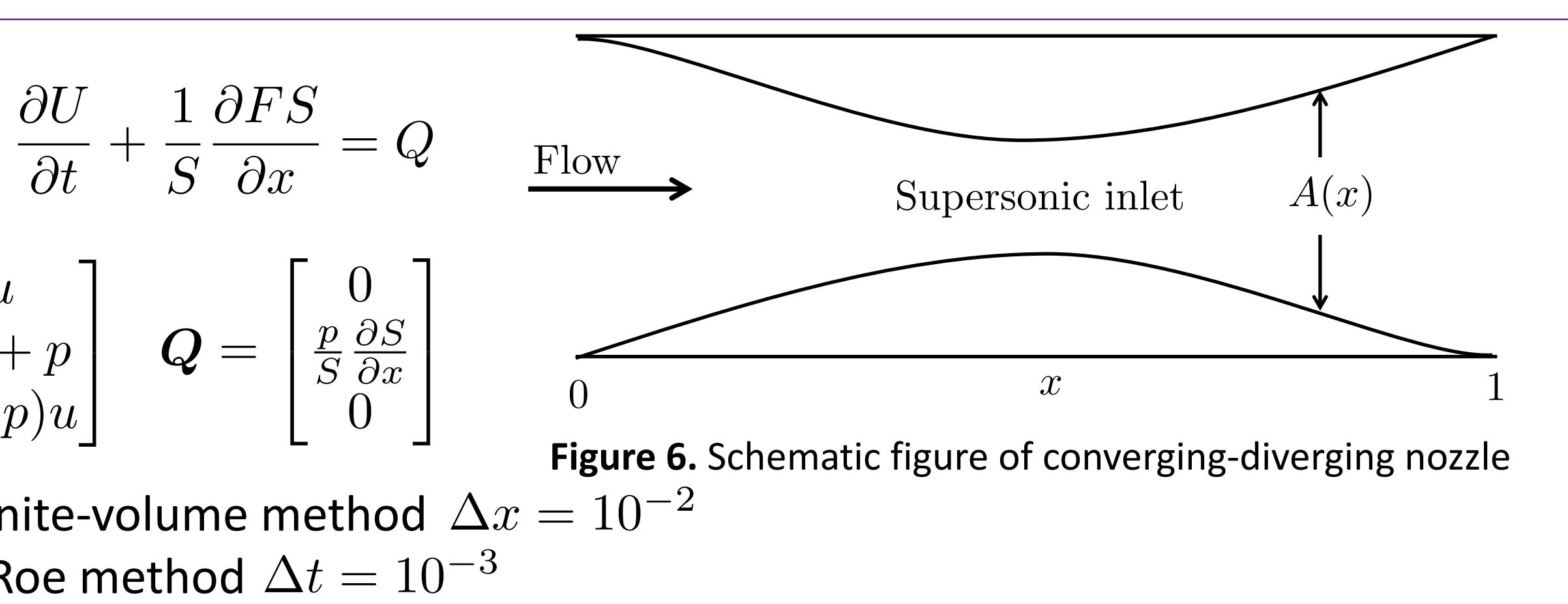
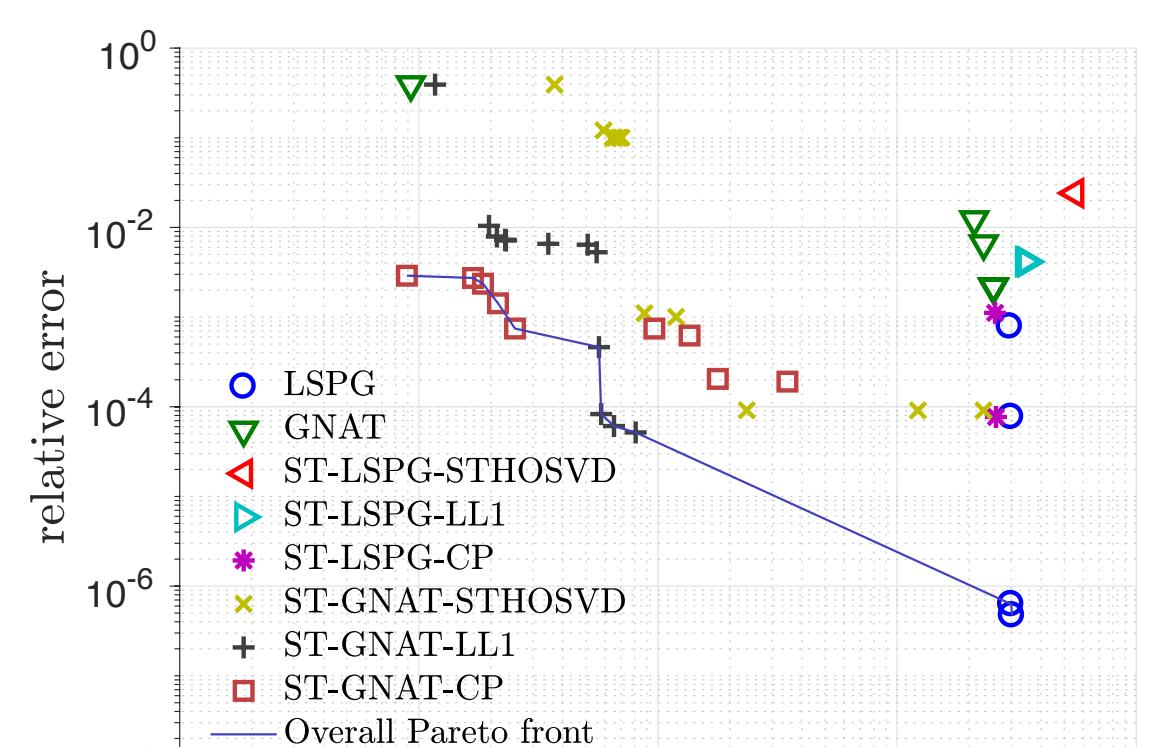


Figure 4. Neutron flux at final time step



Figure 5. Relative error of Energy group 8



### Quasi 1-D Euler equation

- Governing Equation  $\frac{\partial U}{\partial t} + \frac{1}{S} \frac{\partial FS}{\partial x} = Q$
- $U = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix} \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (e+p)u \end{bmatrix} \quad Q = \begin{bmatrix} 0 \\ \frac{p}{S} \frac{\partial S}{\partial x} \\ 0 \end{bmatrix}$
- Spatial discretization: a finite-volume method  $\Delta x = 10^{-2}$
- Time integrator: implicit Roe method  $\Delta t = 10^{-3}$
- Parameter: exit-pressure increase & Mach number at the middle
- Training set:  $\mathcal{D}_{train} = \{1.7 + 0.01i\}_{i=0}^3 \otimes \{1.7, 1.72\}$

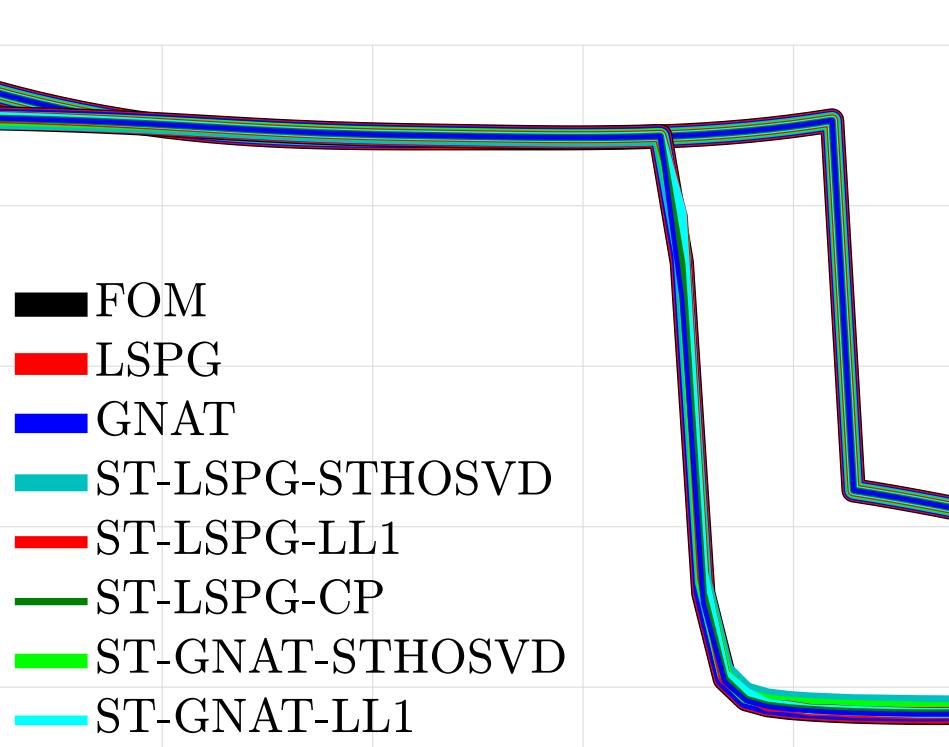


Figure 7. Pareto front,  $\mu = (1.7125, 1.71) \notin \mathcal{D}_{train}$

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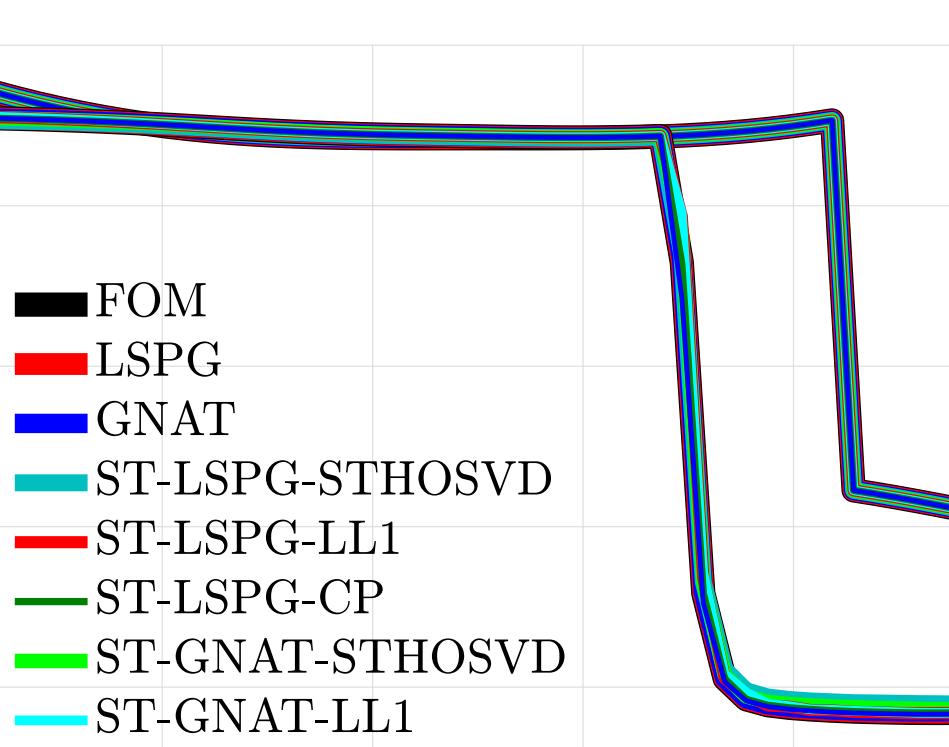


Figure 8.  $n_s = 50$ ,  $n_t = 30$ ,  $n_t^i = 3$   
 GNAT:  $n_z = n_r = 145$   
 GNAT-ST1:  $\bar{n}_s = 120$ ,  $\bar{n}_t = 20$ ,  $n_{r,s} = 150$ ,  $n_{r,t}^i = 10$   
 GNAT-ST2:  $\bar{n}_s = 140$ ,  $\bar{n}_t = 100$ ,  $n_{r,s} = 150$ ,  $n_{r,t}^i = 10$

Method	Rel. Error	Speedup
LSPG	0.048	0.92
GNAT	0.18	0.97
ST-LSPG1-HOSVD	2.32	0.34
ST-LSPG-II1	0.79	0.89
ST-LSPG-CP	0.018	1.2
ST-GNAT-HOSVD	0.94	11
ST-GNAT-II1	1.2	10
ST-GNAT-CP	0.59	19