Multilevel Graph Coarsening with Applications
Benjamin Quiring, Northeastern University and Panayot Vassilevski, CASC, LLNL

Setting

Given a graph $G$ with vertices $V$ and edges $E$:
- Each edge $e = (u, v)$ has a non-weight $w_{uv}$.
- Each vertex $x$ has a (weighted) degree $d_x = \sum_{y \in N(x)} w_{xy}$.

Note: $w_{uv} \geq 0$ is allowed, but $d_x > 0$ must hold.

Abstract

Goal: to generate a hierarchy of coarser graphs that maintain certain properties of the original graph by using pairwise aggregation of strongly connected vertices and maximizing the modularity functional $Q$. After creating the hierarchy, embed the graph in $R^2$ as a multi-layered function.

Results: the aggregation algorithm achieves higher maximum $Q$ values than the state of the art, Lenovo's Algorithm. Additionally, we applied the aggregation to classification and algebraic multigrid methods, forming aggregates that correspond to different versions of the direction of the abstract anisotropic diffusion operator.

Multilevel Graph Coarsening

Goal: Find a "balanced" partition of $G$.

The coarse graph has vertices $|A|$ with the following values:
$$d_A = \sum_{e \in E} \sum_{x \in V} w_{xy}, a_{xy} = \sum_{x \in V} w_{xy}, a_x = \frac{d_A + \sum a_{xy}}{2} \in (0,1).$$

With $T = \sum_{x \in V} a_x$ we maximize the modularity functional $Q$:
$$Q = \frac{1}{T} \sum_{x,y} \left( \frac{a_{xy}}{a_x a_y} - 1 \right) \left( 1 - \frac{a_x a_y}{a_x + a_y} \right).$$

Assign to each edge $e = (x,y)$ a surface-to-center ratio between $\alpha$ and $\beta$.

$$\eta _{xy} = \frac{\eta _{xy} = \alpha a_{xy} (\log a_x + a_y) - (1 - \alpha) (1 - a_x) a_y}{\alpha a_x a_y} \quad (2)$$

Algorithm: Multilevel Graph Coarsening

- Initialize each $A = \{1\}$, $d_A$, $a_{xy}$, and $\eta _{xy}$ for each $e = (x,y)$.
- Take the maximum $\eta _{xy}$, merge aggregates $A$ and $B$ into a new aggregate $C$. Compute $d_C$, $a_C$, and $\eta _{C}$.
- Reconstruct $\eta _{xy}$ and update $Q$. If $Q$ achieved a maximum and has not decreased, create another partition.
- Iterate until there are no more edges, and return the hierarchy of partitions.

Application: Embedding in $R^2$

Using the hierarchy of coarse graphs, embed $G$ in the unit disk $C^2$ for a given $A$.

Given an embedding for the coarsest level graph, move corre-jour to assign coordinates to the finest level vertices.

Algorithm: Embedding in $R^2$

- Initialize $a_x = x_i^2 + x_j^2$, $A = \{1\}$, $i \in A \in E$, $i = 1$, and $x_j \in (-1,1)$ a random number.
- Loop over each $x_j$ and each $A$ and update $a_x$. Sort $A_x$ into $A_x$ and $x_j^2 + x_k^2$ such that $x_j < x_i < x_k$. Then we have:

$$0 < x_j < x_i < x_k$$

For each interval $x \in [x_j, x_k]$, we find the function $A_x(x)$ and construct the form:
$$A_x(x) = \sum_{x \in [x_j, x_k]} \frac{x}{x_j - x_k} + 1.$$

The derivative and second derivative are:
$$\frac{dx}{dx} = \sum_{x \in [x_j, x_k]} \frac{1}{x_j - x_k}, \frac{d^2x}{dx^2} = \sum_{x \in [x_j, x_k]} \frac{1}{(x_j - x_k)^2} > 0.$$

Therefore the equation $\frac{dx}{dx} = 0$ has a unique solution $x = x^*$, in the interval $x \in [x_j, x_k]$.

Acknowledgements

Stephen Golov: https://github.com/glov/suanlapp

References


This work was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.